where it is assumed

$$V = \int_{V_0} v dV_0, \quad K = \int_{V_0} k dV_0. \tag{15}$$

Comparing (1) and (14), the deduction can be made that using the Vernotte–Lykov relationship (4) in place of the Fourier hypothesis (3) would result in the appearance of an additional member in the Biot variational equation. Let us note that this additional member agrees in form (see the expression in the square brackets in the generalized equation (14)) with the corresponding component of the Lagrange equation in analytical mechanics, which takes account of the influence of the kinetic energy of the mechanical system [8]. For $t_r = 0$ and $K = D$ Eq. (14) goes over into the Biot equation (1).

NOTATION

T, temperature; t, time; t_r, relaxation time; x, y, z, spatial coordinates; ρ, c, λ, coefficients of volume density, specific heat, and heat conductivity of the material; q_i, generalized coordinates; the dot above the variables q_i and H denotes differentiation with respect to time; V_0 = const, S_0 = const are the body volume and surface area bounding this volume, respectively.

LITERATURE CITED

1. M. A. Biot, Variational Principles in Heat Transfer, Oxford Univ. Press (1970).
2. P. Vernotte, C. R., 246, No. 22 (1958).
3. A. V. Lykov, Theoretical Principles of Chemical Technology [in Russian], Vol. 1, No. 5 (1967).
4. L. A. Brichkin, Yu. V. Darinskii, and L. M. Pustyl'nikov, Inzh.-Fiz. Zh., 26, No. 3 (1974).
5. B. Coniaux, Rev. Energ. Prim. EPE, 3, No. 1 (1967).
6. N. V. Antonishin, M. A. Geller, and A. L. Parnas, Inzh.-Fiz. Zh., 26, No. 3 (1974).
7. Yu. A. Samoilovich, Inzh.-Fiz. Zh., 29, No. 5 (1975).
8. H. Goldstein, Classical Mechanics, Addison-Wesley (1950).

EFFECT OF SLITS ON THE RESISTANCE OF A CONDUCTING FILM

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Formulas are derived for the resistance offered to a steady or quasisteady current by a conducting film with straight and Π-shaped slits.

One way of changing the resistance of a conducting film used as a resistor is to cut slits in it. A slit is a thin curve along which a conducting layer has been removed, i.e., in slits the conductivity $\sigma = 0$.

The resistance of a film with a slit can be calculated by using the familiar analogy between the hydrodynamics problem of flow past a certain body and the electrostatics problem [1], and also the analogy between the electrostatics problem and the problem of the distribution of a steady or quasisteady current [2].

1. We calculate the change in resistance of a film with a straight slit perpendicular to the lines of flow at infinity (Fig. 1). We denote the width of the film by $2l$ and the length of the slit by 2b. We assume that $l \gg b$ and that the slit is located in the middle of the film so that the perturbing effect of its ends does not extend to the edges of the film. The solution of the corresponding hydrodynamics problem of the flow of a fluid past a plate in an infinite medium is given in [1]. For the problem under consideration, we write the conformal mapping function in the form

$$W(z) = -iE_{y\infty}\sqrt{z^2 - b^2}, \tag{1}$$

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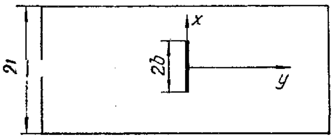

Fig. 1

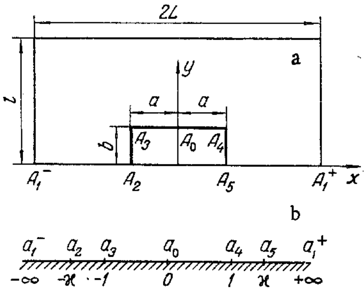

Fig. 2

Fig. 1. Film with a straight slit perpendicular to the lines of flow.

Fig. 2. a) П-shaped slit at edge of film; b) conformal mapping of П-shaped slit onto upper half plane.

where $z = x + iy$ and $E_{y\infty}$ is the electric field intensity at infinity. The current flows along the film, and $E_{x\infty} = 0$. In accord with [3], we write

$$W(z) = U(x, y) + iV(x, y), \tag{2}$$

where $U(x, y)$ is the potential function and $V(x, y)$ is the stream function. Then the resistance R of the film bounded by the equipotential lines U_1 and U_2 and lines of force V_1 and V_2 is

$$R = \zeta \frac{|U_2 - U_1|}{|V_2 - V_1|}; \tag{3}$$

where ζ is the resistance of a unit square of the film. After finding functions $U(x, y)$ and $V(x, y)$ from (1) and (2) and using the fact that the dimension of the film along the y axis is very much larger than $2l$, we obtain from (3)

$$R = \zeta \frac{|y_1| + |y_2|}{2\sqrt{l^2 - b^2}}, \tag{4}$$

where $y = y_1$ for $y > 0$ and $y = y_2$ for $y < 0$. For $l \gg b$ a simple expression can be found from (4) for the change in resistance of the film ΔR

$$\Delta R = R - R_0 \approx R_0 \frac{b^2}{2l^2}, \tag{5}$$

where R_0 is the resistance of the film without a slit.

By using the method described it is easy to treat the problem of a slit at the edge of a film, i.e., a film cut into two halves (Fig. 1) along the y axis. The conformal mapping function for this case is known [1]. Appropriate calculations yield the same result as that given by Eq. (5), i.e., the relative change in resistance is the same as before. Physically this is understandable since the film in Fig. 1 can be regarded as two films with slits along their edges connected in parallel along the y axis.

2. The result obtained is understandable since the change in resistance of the film is approximately proportional to b^2. In addition, it is desirable to have a smoother change in resistance. To accomplish this, we consider the П-shaped slit shown in Fig. 2a, where l is the width of the film, $2L$ is its length, b is the height of the slit, and $2a$ is its length. It is convenient to perform the calculation in the dimensionless variables $h = l/a$, $\mathscr{L} = L/a$, and $\varepsilon = b/a$. Assuming

$$h \gg \varepsilon, \quad \mathscr{L} \gg 1, \tag{6}$$

we map the pentagon $A_1A_2A_3A_4A_5$ in the complex W plane (Fig. 2a) conformally onto the upper half of the complex z plane (Fig. 2b).

The transformation $W = W(z)$ which produces the required confirmal mapping can be determined by using the principle of symmetry and the Schwarz–Christoffel integral [1]. For the problem under consideration this integral has the form

$$W(z) = C_1 \int_0^z \sqrt{\frac{1 - z^2}{\varkappa^2 - z^2}}\, dz + C_2, \tag{7}$$

or

$$W(z) = C_1 k G(\varphi, k) + C_2, \tag{8}$$

where $\sin\varphi = z$, $\varkappa = k^{-1}$, and k is the modulus of the elliptic integrals; $G(\varphi, k)$ denotes

$$G(\varphi, k) \equiv F(\varphi, k) - D(\varphi, k),$$
$$D(\varphi, k) = \frac{F(\varphi, k) - E(\varphi, k)}{k^2}, \quad D = \frac{K - E}{k^2}. \tag{9}$$

From the correspondence between the points a_0 and A_0, and from (8) we obtain

$$C_2 = i\varepsilon. \tag{10}$$

Using the pairs of points a_4, A_4 and a_5, A_5, and (10) we obtain

$$C_1 = (kG)^{-1}, \tag{11}$$

$$\int_1^{\varkappa} \sqrt{\frac{z^2 - 1}{\varkappa^2 - z^2}}\, dz = \varepsilon G k, \tag{12}$$

where $G \equiv K - D$.

Substituting (10), (11), and (12) into (7), we find

$$W(z) = \frac{G(\varphi, k)}{G} + i\varepsilon. \tag{13}$$

To calculate the resistance it is necessary to determine certain values of $z = z(W)$. From considerations of symmetry $a_1^+ = -a_1^- \equiv m$. The value of m can be found after taking the value of $W(z)$ at $z = m$. After integration, we obtain

$$\mathcal{L} kG = kG + \frac{\varkappa^2 - 1}{\varkappa} F(\mu, k) - \varkappa E(\mu, k) + m\sqrt{\frac{m^2 - \varkappa^2}{m^2 - 1}}, \tag{14}$$

where

$$\mu \equiv \arcsin\sqrt{\frac{m^2 - \varkappa^2}{m^2 - 1}}.$$

Hence, restricting ourselves to terms on the order of m^{-2},

$$m \approx \mathcal{L} kG + \frac{\varkappa^2 - 1}{2\mathcal{L} kG}. \tag{15}$$

Further, let us find H, the image of the distance between the points corresponding to $z = 0$ and $\mathrm{Im}\, z = i\mathcal{L}$ in the z plane. Taking account of (10) and (11) we have from (8)

$$i(h - \varepsilon) kG = \int_0^{iH} \sqrt{\frac{1 - z^2}{\varkappa^2 - z^2}}\, dz, \tag{16}$$

from which, after integration along a segment of the imaginary axis

$$(h - \varepsilon) kG \approx kF(\varkappa/H, k') - \varkappa E(\varkappa/H, k') + H, \tag{17}$$

where $k' = \sqrt{1 - k^2}$ is the complementary modulus of the elliptic integral. Using the relation [4]

$$E(\varkappa/H, k') \approx F(\varkappa/H, k') + O(H^{-3}), \tag{18}$$

we find

$$H \approx (h - \varepsilon) kG. \tag{19}$$

Remembering that the resistance R of the film with a slit is

$$R = \zeta \frac{a_1^+ - a_1^-}{H}, \tag{20}$$

and without a slit is given by Eq. (4) for $b = 0$, we find

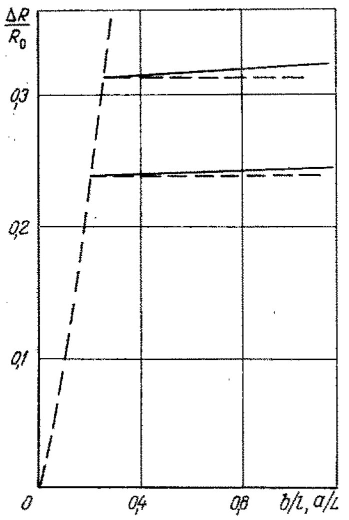

Fig. 3. Relative change of film resistance as a function of size of П-shaped slit. The sharp bend in the curve occurs at the beginning of the horizontal slit.

$$\frac{R}{R_0} = \frac{1}{1-\frac{\varepsilon}{h}}\left[1+\frac{\varkappa^2-1}{2(\mathscr{L}kG)^2}\right] \tag{21}$$

We find $\varkappa$ from (12), and using the fact that $\varepsilon \ll 1$, we obtain

$$\varkappa^2 \approx 1+\frac{4}{\pi}\varepsilon+\frac{2}{\pi^2}\left(2\ln\frac{\varepsilon}{4\pi}+5\right)\varepsilon^2, \tag{22}$$

$$k^2 = 1-k'^2 \approx 1-\frac{4}{\pi}\varepsilon-\frac{2}{\pi^2}\left(2\ln\frac{\varepsilon}{4\pi}-3\right)\varepsilon^2,$$

which is accurate to terms on the order of ε^2. In addition G can be found from the expansion [4]

$$G \approx 1+\frac{3-2\Lambda}{4}k'^2+\frac{51-36\Lambda}{2}k'^4, \tag{23}$$

where $\Lambda = -\frac{1}{2}\ln(k'^2/16)$. Using (22) and (23) we convert to dimensional variables in (21) and find the change in resistance of the film ΔR

$$\Delta R = R - R_0 \approx R_0\left(\frac{b}{l}+\frac{b^2}{l^2}+\frac{2}{\pi}\frac{ab^2}{L^2 l}\right). \tag{24}$$

In practice the resistance of a film is changed by making a Г-shaped cut. It is known from experiment that ΔR varies parabolically with b and linearly with a.

The calculation of the resistance of a film with a Г-shaped slit is difficult to perform analytically, and so far has not been accomplished. It is clear from the hydrodynamic analogy that the patterns of flow past Г-shaped and П-shaped slits for $a \gg$ b are nearly the same. Therefore Eq. (24) can be used to calculate the Г-shaped slit with an accuracy which increases with decreasing ε. This is illustrated in Fig. 3 which shows $\Delta R/R_0$ as a function of the size of a П-shaped slit. The pattern obtained is in good agreement with the analogous experimental curve for a Г-shaped slit.

In conclusion, we note that the conformal mapping function found above can be used to solve the problem of a thermal field in an analogous system by replacing the slit by a thermally insulating cut. Such problems can be solved rather simply by conformal mapping [5, 6], but are more difficult with singular integral equations [7]. In [6] there is also considered a rectangular notch in an infinite strip of a uniform sheet, but the heat flux is directed across the strip and the Schwarz formula for a half plane [1] is used in the calculation.

NOTATION

σ, conductivity; $z = x + iy$, complex variable; i, imaginary unit; W(z), function of complex variable; E_{y_∞}, electric field intensity along y axis at infinity; U(x, y), potential function. V(x, y), stream function; R, resistance of film with slit; ζ, resistance of unit square of film; R_0, resistance of film without slit; ΔR, change in resistance; $h = l/a$, $\mathscr{L} = L/a$, $\varepsilon = b/a$, dimensionless variables; A_i, vertices of pentagon in complex W plane; a_i, images of vertices of pentagon; C_1, C_2, constants in Schwarz–Christoffel integral; $F(\varphi, k)$, $E(\varphi, k)$, elliptic integrals of first and second kind; k, modulus of elliptic integrals; $\sin\varphi = z$; $\varkappa = k^{-1}$; K, complete elliptic integral of first kind; E, complete elliptic integral of second kind; k', complementary modulus of elliptic integral; $O(H^{-3})$, infinitesimal of third order.

LITERATURE CITED

1. M. A. Lavrent'ev and B. V. Shabat, Methods of the Theory of Functions of a Complex Variable [in Russian], Nauka, Moscow (1973).
2. Ya. I. Frenkel', Electrostatics and Electrodynamics [in Russian], Vol. 2, ONTI, Moscow–Leningrad (1935).
3. W. Smythe, Static and Dynamic Electricity, McGraw-Hill, New York (1950).
4. I. S. Gradshteyn and I. M. Ryshik, Tables of Integrals, Series, and Products, Academic Press, New York (1965).
5. N. V. Pal'tsun and A. G. Gorban', Inzh.-Fiz. Zh., 28, No. 5 (1975).
6. G. I. Vernikov, Inzh.-Fiz. Zh., 28, No. 6 (1975).
7. M. P. Savruk and I. F. Soltys, Inzh.-Fiz. Zh., 32, No. 2 (1977).